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Non-Gaussian Fluctuations of Local Lyapunov Exponents at Intermittency

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In intermittent dynamical systems, the distributions of local Lyapunov exponents are markedly non-Gaussian and tend to be asymmetric and fat-tailed. A comparative analysis of the different time-scales in intermittency provides a heuristic explanation for the origin of the exponential tails, for which we also obtain an analytic expression deriving from a more quantitative theory. Application is made to several examples of discrete dynamical systems displaying intermittent dynamics.

KEY WORDS: Intermittency; finite time Lyapunov exponents; exponential tails.

1. INTRODUCTION

A variety of diverse dynamical behaviors finds itself classified under the rubric of the term "intermittency."[1,2] The characteristic feature of such motion is a switching between two distinct states. The classic Type I, II, and III intermittencies, first described by Pomeau and Manneville[1] involve the motion alternating between laminar and chaotic states, the distinctions between these types arising in the scaling exponents for the duration spent in one or the other state as a parameter is varied. Other forms of intermittency are known (crisis induced,[3] Type V,[4] on-off,[5] in-out,[6] etc.), and each of these differ in either the nature of the two states between which the motion switches, or in the nature of the scaling behavior and exponents or both. In Hamiltonian systems, the dynamics can also display intermittency if the motion is trapped in the vicinity of KAM islands in the mixed phase space in quasi-integrable systems;[7] similar effects have been shown also to give rise to long-time tails in correlation functions in chaotic systems.[8]

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Owing to this alternation between (at least) two dynamical state over long periods of time, global quantities often provide a poor characterization of intermittent dynamics.[9] A case in point is the Lyapunov exponent in Type-I intermittency.[2] Even though such motion is chaotic (in the sense of showing sensitivity to initial conditions), the Lyapunov exponent can be close to zero if the duration of the laminar phase greatly exceeds that of the chaotic bursts. Local quantities, on the other hand, provide a more detailed description of the dynamics since they probe the motion on short enough time-scales to distinguish the different states that the system passes through. In such situations, distributions of local Lyapunov exponents (LLEs), namely the value of the Lyapunov exponents over finite segments of a trajectory, provide a better probe of the underlying nonuniform attractor.[10, 11] In particular, the local Lyapunov exponent can be negative even when the global Lyapunov exponent is positive (as on a typical chaotic attractor) or vice versa (on strange nonchaotic attractors).[12]

This paper is concerned with the study of intermittent dynamics in terms of the distribution of LLEs. We show that in intermittent systems, there are significant finite-size effects that lead to a departure from the general theory[11, 13] for LLEs in typical chaotic dynamical systems. Thus, the LLE distributions are markedly non-Gaussian, regardless of the actual type of intermittency, being characterized by exponential asymmetric tails.

Understanding the origin of these tails is the main focus of the present study. Indeed, in a variety of other contexts wherein intermittency plays a major role (as for example at the onset of turbulence), it is well-known[14] that quantities such as velocity distributions show exponential behavior. We provide a heuristic explanation for the existence of such fat-tailed distributions, and also give a phenomenological theory which appears to satisfactorily account for the exponential tails. Details of the theory are given in Section III of this paper. The dynamical systems considered here are mainly discrete maps in 1-dimension which have been extensively studied in the context of intermittency. In the next section we define the various quantities such as Lyapunov exponents, LLEs and their distributions, and examples of these quantities at intermittencies. We conclude in Section IV with a summary and discussion of our results.

2. INTERMITTENT DYNAMICAL SYSTEMS

2.1. Lyapunov Exponents and their Distributions

Lyapunov exponents measure the rates at which volume elements in phase space expand or contract along an orbit and provide a classification of the nature of the dynamics.[15] For an ergodic system in d dimensions, there are d Lyapunov exponents, which are independent of almost every initial conditions on the attractor except for a measure zero set of points.[16] Integrable conservative systems have all Lyapunov exponents equal to zero. If one or more exponent is positive, the dynamics is chaotic, and nearby trajectories diverge exponentially from each other.[17] When the largest Lyapunov exponent is negative, then nearby trajectories converge, and the dynamics is stable.[18]

The dynamical systems we consider here are 1-dimensional mappings of the interval onto itself, defined by

$$x_{n+1} = f(x_n) \tag{1}$$

where $f(x)$ is a differentiable or piecewise differentiable function. The Lyapunov exponent for this system, Λ, is given by

$$\Lambda = \lim_{N \to \infty} \lambda_N = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ln \left| \frac{df(x_i)}{dx} \right| \tag{2}$$

where λ_N is the N-step finite time local Lyapunov exponent. Note that in 1-dimension LLE's are simply sum of local stretch factors, given by $\ln |f'(x)|$. For chaotic dynamics, instantaneous LLEs (namely the case $N = 1$) can be considered as random independent and identically distributed variables. LLEs are fluctuating quantities and depend on initial conditions. Their probability density, however, is stationary, with respect to the invariant measure for the dynamics and is defined by

$$P(N, \lambda)\, d\lambda = \text{probability that } \lambda_N \text{ lies between } \lambda \text{ and } \lambda + d\lambda. \tag{3}$$

For hyperbolic systems, a general argument[11, 17] following the central limit theorem[19] gives

$$P(N, \lambda) \sim \exp(-NG(\lambda)) \tag{4}$$

with $G(\lambda)$ having a quadratic maximum. Note that this does not constrain the form of the distribution away from the maximum, and a variety of behaviours in the tail of the distribution can be consistent with the limiting behaviour as $N \to \infty$, when the distribution will tend to a δ-function centered on Λ

$$P(N, \lambda) \to \delta(\lambda - \Lambda). \tag{5}$$

There are, however, departures from this universal behavior when the dynamics is nonhyperbolic. One prominent example is that of the quadratic logistic map[9, 20]

$$x_{n+1} = \alpha x_n(1 - x_n) \tag{6}$$

at the Ulam point, $\alpha = 4$ when $\Lambda = \ln 2$, $P(N, \lambda)$ has a cusp and the form of the distribution is approximated by[9]

$$P(N, \lambda) = \frac{N}{\pi} \frac{\exp(-N\,|\lambda - \Lambda|)}{[1 - \exp(-2N\,|\lambda - \Lambda|)]^{1/2}}. \tag{7}$$

2.2. Examples

The form of $P(N, \lambda)$ at intermittency is best illustrated by examples. Type I intermittency[1] in the logistic map, Eq. (6) occurs immediately prior to the tangent bifurcation at $\alpha = \alpha_t = 1 + \sqrt{8}$. We examine the dynamics at

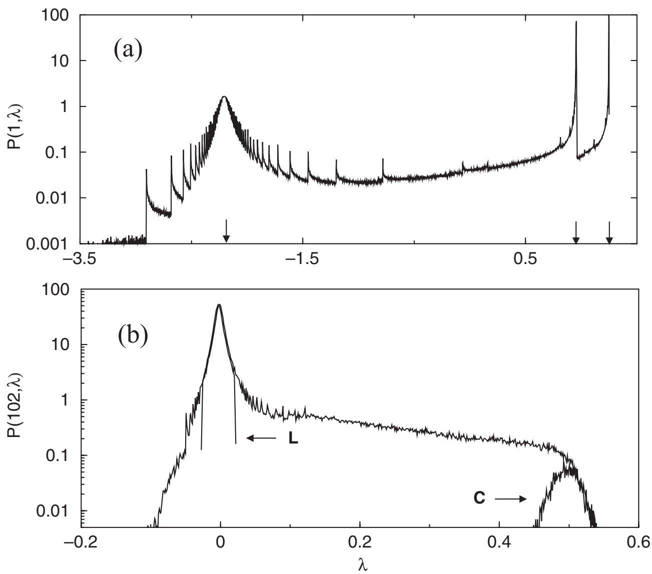

Fig. 1. The distribution of finite time Lyapunov exponents for the logistic map near the tangent bifurcation at $\alpha_t = 1 + \sqrt{8}$. (a) The distribution of stretch exponents, namely λ_1 at $\alpha = \alpha_t - 10^{-5}$. The arrows indicate the values of stretch exponents corresponding to the period three orbit at α_t. (b) The distribution of LLEs for $N = 102$. The curve marked L gives the (Gaussian) distribution of those LLEs which come entirely from the laminar phase and that marked C is the distribution of LLEs entirely within the chaotic region.

$\alpha = \alpha_t - \epsilon$, $\epsilon = 10^{-5}$. The stretch exponents, λ_1, have the density shown in Fig. 1(a). (At $\alpha > \alpha_t$, when the dynamics is on the period 3 orbit, the density is a set of δ-functions at the three values indicated by arrows on the axis in Fig. 1(a).) By averaging these stretch exponents over a finite portion of a trajectory, one obtains finite-time Lyapunov exponents. Shown in Fig. 1(b) is the LLE distribution at the same value of $\alpha = \alpha_t - \epsilon$. Even for this relatively large value of N, the distribution of $P(N, \lambda)$ is highly asymmetric with a distinct exponential tail. As shown in ref. 9, the total density can be decomposed into portions coming from trajectory segments that are entirely in the laminar part or entirely in the chaotic burst (the dotted lines in Fig. 1(b)), with the exponential portion of the distribution arising from trajectory segments that have both dynamics.

A similar asymmetric distribution arises from the dynamics of the cusp map,[21]

$$f(x) = 1 - 2\,|x|^{1/2} \qquad x \in [-1, 1] \tag{8}$$

which has a marginal fixed point at $x = -1$. The dynamics is largely chaotic except when the trajectory is trapped in the vicinity of the marginal fixed point at $x = -1$, and therefore the non-Gaussian tail of the distribution of LLEs is to the left of the mean; see Fig. 2(b).

One can formally build up the distribution $P(N, \lambda)$ if the invariant density is known exactly.[9] For the logistic map, the invariant density is not known for parameter value α_t, though for the cusp map it is[21] $\rho(x) = (1-x)/2$. One map which shows intermittency and for which the invariant density is given exactly by

$$\rho(x) = \frac{x^{\alpha}(1-x)^{\beta}}{B(\alpha+1, \beta+1)} \tag{9}$$

where $\alpha, \beta > -1$, and B is the beta function,

$$B(\alpha+1, \beta+1) = \int_0^1 x^{\alpha}(1-x)^{\beta}\, dx. \tag{10}$$

can be obtained by inversion.[22] For parameters $\alpha = -0.62$ and $\beta = -0.4$, the implicit map $f(x)$ has intermittent dynamics and its LLE distribution shown in Fig. 2(a) also has the characteristic exponential tail.

In on-off intermittency,[5] the "off" state is nearly constant. The dynamics typically is in this state for long durations, moving to the "on" state when it displays intermittent bursts, and returning to the off state rapidly. The basic requirement for on-off intermittency is that the system possess an

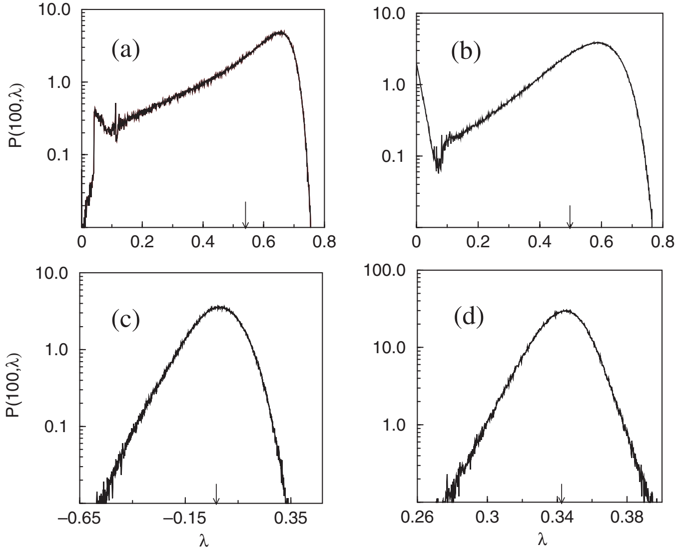

Fig. 2. The characteristic probability densities for intermittent dynamics in (a) the beta map at parameter value $\alpha = -0.62$ and $\beta = -0.4$ (Eq. (9)), (b) the cusp map (Eq. (8)), (c) the driven logistic map at parameter value $a = 2.8$ (Eq. (11)) and (d) the logistic map at the band-merging crisis (Eq. (6)), at $\alpha = 3.6785735104284$. The arrow in each figure points to the value of corresponding asymptotic Lyapunov exponent.

invariant subspace which can be made unstable;[23] an example is the driven logistic map[24]

$$y_{n+1} = z_n y_n (1 - y_n) \tag{11}$$

where $z_n = ax_n$ and x_n is a uniform random variable in the interval $[0,1]$. For $a = 2.8$, this map shows on-off intermittent behavior, and the corresponding distribution of LLEs (Fig. 2(c)) also has the characteristic exponential non-Gaussian tail.

The final example of intermittency shown here in Fig. 2(d) is that which occurs in the neighborhood of crisis phenomena.[17] Recall that at crises, there is an abrupt change in the phase space volume; this can be caused by collision of a chaotic attractor with an unstable periodic orbit at the so-called band-merging crisis. An example of such dynamics obtains in the logistic map, Eq. (6) at $\alpha \approx 3.678\ldots$[25] Here both the states between which the dynamics switches have chaotic dynamics, and again the resulting LLE distribution has an exponential tail.

3. MODELING

When the invariant density is known exactly (as in some of the examples above), then it is in principle possible to derive the exact distribution $P(N, \lambda)$ though the resulting algebra is not always very transparent. Given a map $f(x)$ and its invariant density $\rho(x)$ the distribution for the case $N=1$, namely the stretch exponent, is obtained by inverting the relationship

$$P(1, \lambda) = \rho(x) \left| \frac{dx}{d\lambda} \right| = \rho(x) \, |f(x)'|. \tag{12}$$

For $N > 1$, $P(N, \lambda)$ must be calculated recursively. Since λ_N is known in terms of x implicitly, it will therefore satisfy an equation of the form

$$\exp N\lambda_N = G(x) \equiv \prod_{i=1}^{N} \frac{d(f^{(i)}(x))}{dx}. \tag{13}$$

The functional form of the resulting G will depend on the map under consideration; for polynomial f, G is polynomial as well. Summing over all real roots of the equation $\exp(N\lambda) - G(x) = 0$, one finally obtains[9] the distribution as

$$P(N, \lambda) = N \exp(N\lambda) \sum_{\text{roots}} \frac{\rho(x)}{|G'(x)|}. \tag{14}$$

This program is limited by the necessity of knowing $\rho(x)$ analytically.

For maps of the form $1 - 2\,|x|^z$, where $\frac{1}{2} \geqslant z \geqslant 1$, $x = -1$ is a marginal fixed point, so that the dynamics is intermittent. The invariant measure is not known analytically except for the case $z = \frac{1}{2}$, when it is given by $\rho(x) = (1-x)/2$. As can be easily seen, the 1-step LLE has the distribution $P(1, \lambda) \sim \exp(-2\lambda)$. Using Eq. (14), we obtain the N-step LLE distribution for low values of N by following the procedure outlined above; this is shown in Fig. 6 for $N = 5$. Although this approach is exact, for actual application the necessity of knowing large number of roots for higher values of N limit its utility. In the general case, the distribution for the 1-step Lyapunov exponent appears (numerically) to have the form $P(1, \lambda) \sim \exp(-4z\lambda)$, which suggests that a similar analysis will also give rise to exponential tails for larger N.

Below we present a phenomenological theory of LLE's at intermittency. In any computation of a finite time Lyapunov exponent λ_N is computed as the average of a set of N stretch exponents (λ_1's), some N_l of which may come from laminar motion and $N_c = N - N_l$ from chaotic motion.[26] If the entire portion of the trajectory falls within the laminar or chaotic state, the resulting distribution of such LLE's is Gaussian, viz.

$$P(\lambda, \mu_\alpha, \sigma_\alpha) = \frac{1}{\sqrt{2\pi\sigma_\alpha^2}} \exp\left[-\frac{(\lambda - \mu_\alpha)^2}{2\sigma_\alpha^2} \right] \tag{15}$$

where the subscript $\alpha = l, c$ labels the distinct state (laminar or chaotic) involved in the intermittency.

The distributions of the stretch exponents, the one-step LLE is itself not Gaussian: see Fig. 1(a) for a typical case. For the case of λ_N arising from N_l steps in the laminar and N_c steps in the chaotic region (as in the trajectory illustrated in Fig. 3), we make the assumption that the resulting distribution is also Gaussian

$$P(n_l, n_c, \lambda) = \frac{1}{\sqrt{2\pi\sigma_{\text{eff}}}} \exp\left[-\frac{(\lambda - \mu_{\text{eff}})^2}{2\sigma_{\text{eff}}} \right] \tag{16}$$

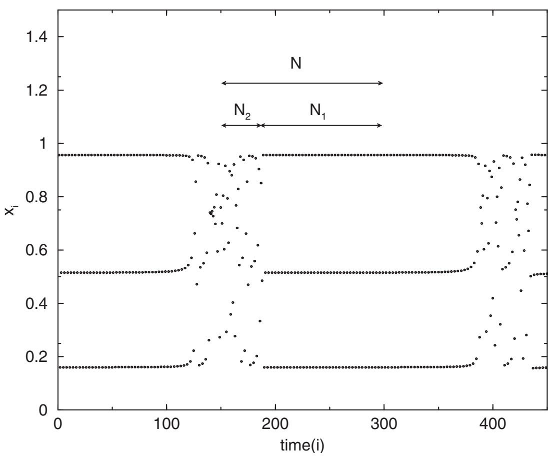

Fig. 3. Time series for typical intermittent dynamics. Any randomly chosen part of the trajectory of length N will be composed of parts coming from the laminar phase (N_1) and parts coming from chaotic phase (N_2).

but with weighted mean and variance

$$\mu_{\text{eff}} = n_l \mu_l + n_c \mu_c \tag{17}$$

$$\sigma_{\text{eff}} = n_l \sigma_l^{\,2} + n_c \sigma_c^{\,2} \tag{18}$$

where $n_\alpha = N_\alpha / N$. The total distribution form for the intermittent dynamics comes from the composition of different Gaussian distributions given by Eq. (16) with n_l varying from 0 to 1. So the final expression for $P(N, \lambda)$ is given by

$$P(N, \lambda) = \sum_{n_l = 0}^{1} c(n_l, n_c)\; P(n_l, n_c, \lambda) \tag{19}$$

where the coefficient $c(n_l, n_c)$ is

$$c(n_l, n_c) = \lim_{\mathcal{N}(N) \to \infty} \frac{\mathcal{N}(n_l)}{\mathcal{N}(N)} \tag{20}$$

$\mathcal{N}(n_l)$ being the number of times a given n_l is realized in the dynamics, and $\mathcal{N}(N)$ is the total number of samples taken, with

$$\sum_{n_l = 0}^{1} c(n_l, n_c) = 1 \tag{21}$$

to preserve normalization.

In the limit $N \to \infty$, $P(N, \lambda)$ does indeed asymptote to a Dirac δ-function although this is not immediately obvious from Eqs. (16) and (15). With increasing N, the variances, σ_α decrease sharply, as, therefore, do the σ_{eff}. Further, the support of $c(n_l, n_c)$ also decreases sharply, and one can (experimentally) verify that the resulting distribution is narrower and increasingly sharply peaked with increasing N. A plot of $c(n_l, n_c)$ for the case of the logistic map at Type I intermittency is shown in Fig. 4(a).

The above formalism can be tested on the examples of intermittent dynamics shown in Section II. Shown in Fig. 4(b) is the predicted distribution (the dashed line) obtained from the formula Eq. (19). Appropriate values of σ_l, σ_c, μ_l, and μ_c were obtained for the case of intermittency at $\alpha = 3.82842$ in the logistic map, for N$=$102. As can be seen the agreement is good, particularly in the region of the exponential tail.

This general procedure can be carried out for the other cases as well. Comparisons are made (shown in Figs. 5(a)–(d) respectively) for the

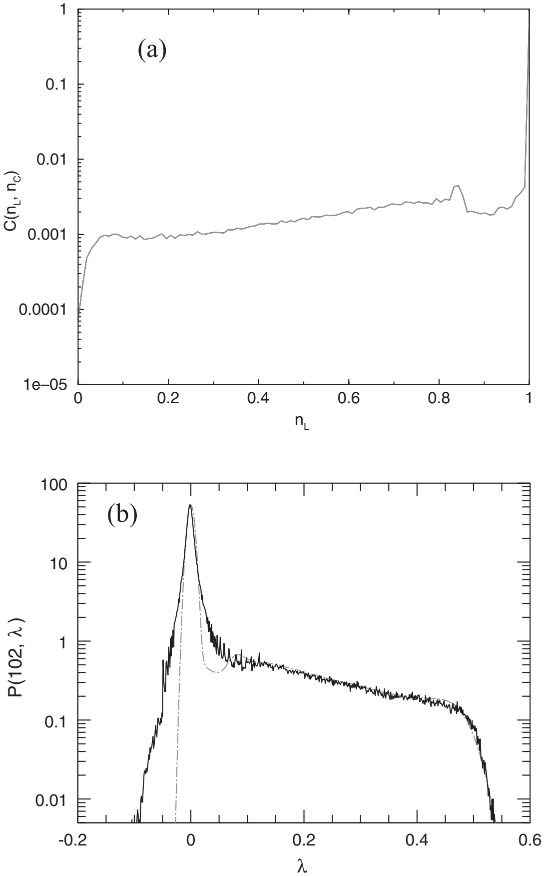

Fig. 4. (a) Plot of $c(n_l, n_c)$ versus n_l for the logistic map. (b) Numerical results (solid lines) compared with the analytic expression obtained from Eq. (19) (dotted lines).

distributions for the beta map at parameter value $\alpha = -0.62$ and $\beta = -0.4$, for the cusp map, as well as for the cases of on-off and crisis induced intermittencies.

4. DISCUSSION

The contrasting information available from global and local indicators of the dynamics is most strongly apparent in intermittent systems. Global quantities are, typically, slowly convergent, and the extreme variability of

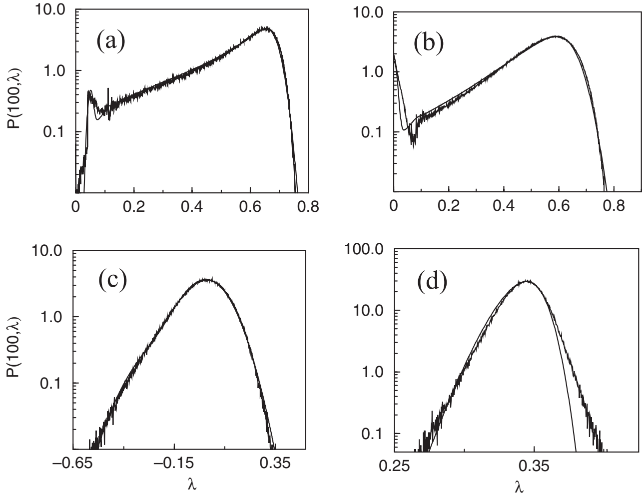

Fig. 5. Comparison of numerical densities and that obtained from Eq. (19) for intermittent dynamics (a) in the beta map (at $\alpha = -0.62$ and $\beta = -0.4$) (Eq. (9)), (b) the cusp map (Eq. (8)), (c) the driven logistic map at $a = 2.8$ (Eq. (11)) and (d) at band-merging crisis in the logistic map (Eq. (6)) at $\alpha = 3.6785735104284$.

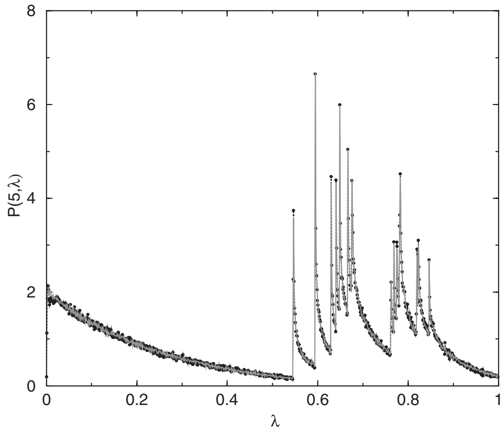

Fig. 6. Comparison of numerical density (dots) and the analytic expression Eq. (14) (solid line) for LLEs in the cusp map (Eq. (8)), for $N = 5$.

the underlying attractors are best described by local quantities which offer a more detailed picture of such systems.

In the present paper we have examined the distribution of local Lyapunov exponents for intermittent dynamical systems. These offer a more detailed description of the dynamics, and are also very relevant in treating experimental data since by its very nature, any measured exponent is a finite-time quantity.[27] We find that all intermittent systems have characteristically asymmetric distributions which are markedly non-Gaussian and with a distinct exponential tail. We give a heuristic explanation for the origin of these tails, and demonstrate the validity of this theory for a number of different examples of intermittency, ranging from Type I, where the motion switches from laminar to chaotic dynamics, to crisis-induced intermittency, where the motion switches between two chaotic states.

The form of the asymmetric and fat-tailed LLE distribution is suggestive of extreme-value statistics[28] although the justification for using such a form is not very clear. A plausible argument for the origin of extreme value statistics in the case where the intermittency switches between a laminar and a chaotic state is as follows. Consider a symbolic encoding of the dynamics in a binary alphabet, with 0 representing the laminar state and 1 the chaotic. Then a finite-time segment of the orbit (a "word") is a binary string of 0's and 1's, with most words having a larger proportion of 0's. If the segment is completely confined to the laminar state, the corresponding word is entirely made of 0's, and the LLE will necessarily be clustered around the mean. If the segment includes a chaotic burst, then the word has some 1's as well, and the value of the LLE must necessarily be much larger. In a distribution this would be an extreme value, and hence the tail of the LLE distribution displays extreme-value statistics. The symbolic encoding also permits the computation of Renyi entropies for the intermittent dynamics,[21] and we have observed that there is a dynamical phase transition in the spectrum of entropies in all cases of intermittency.[29]

Very recently Tribelsky[30] has obtained probability distributions for finite sums of random, arbitrarily correlated variables. As shown there[30] correlations generically give rise to fat-tailed distributions, and thus there is some overlap in the regime of applicability with the present work. The correlations that obtain from intermittent dynamics have not been explicitly considered, but it may be possible to treat this case within the same framework. In addition to local Lyapunov exponents, a number of derived quantities in intermittent dynamical systems also show distributions with exponential tails.[31, 32] The generality of exponential tails in a number quantities in dynamical phenomena known to be intermittent such as the onset of turbulence, for instance[14] could be indicative of the fact that the underlying mechanisms in these phenomena share some commonality.

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